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PROBLEMS WITH THE TOEPLITZ AND QUASI-TOEPLITZ MATRICES THAT APPEAR IN THE STATIC STABILITY ANALYSIS OF ELASTIC SYSTEMS†

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Three stability problems are considered for layered elastic systems. The problems can be treated completely analytically. These systems are packs consisting of arbitrary numbers of identical infinite plates attached by identical layers of a soft medium modelled as a Winkler foundation. The problems differ in the nature of the boundary conditions assigned to the faces of the packets. Each face can be either free, or attached to a non-deformable support. A cylindrical corrugation is sought everywhere. In all cases the critical force, the whole load spectrum and all the forms of stability loss are given by simple explicit formulae.

1. THE TOEPLITZ MATRIX CASE

We will consider the plane formulation of the problem of the stability of plates in a packet, shown in Fig. 1(a). The packet consists of n plates of thickness h, each one made up of a stiff material (with shear modulus G and Poisson's ratio ν), attached along their faces to $n+1$ layers of thickness H of a flexible material (with elasticity characteristics G_p and ν_p). The top and bottom soft layers are attached to non-deformable casings. The x axis of a Cartesian system of coordinates is directed towards us, the y axis to the right and the z axis upwards. The width is the dimension of the packet along x, the length is the dimension along y, and the height is the dimension along z. A compressive load of magnitude T per unit width acts along the y axis on each plate. The plates are numbered from the bottom upwards with indices $i = 1, \ldots, n$. The indices $i = 0, n+1$ are assigned to the casings, if necessary. The cylindrical stiffness of each plate is given by the formula [1–3]

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

The plate deflection w_i as a result of the load T is governed by the equation [1–3]

$$Dw_i'''' + Tw_i'' = q_i, \quad i = 1, \ldots, n \tag{1.1}$$

(where the primes denote differentiation with respect to y).

The transverse load on the ith plate, q_i, is composed of loads q_i^+ and q_i^- from the soft layers lying above and below the plate. We adopt the Winkler hypothesis $q = -\beta w$ for the soft layers, with rigidity coefficient β. Here, naturally, $q_i^+ = -q_{i+1}^-$: the soft layers do not "redistribute" the pressure along the y axis. We have

$$q_i = q_i^+ + q_i^-, \quad q_i^+ = -\beta(w_i - w_{i+1}), \quad q_i^- = -\beta(w_i - w_{i-1}) \tag{1.2}$$

Note that the loads are proportional to the *difference* of the deflections of neighbouring plates, and the rigidity of the medium to small bending is given by the formula

$$\beta = \frac{G_p}{H(1-\nu_p)} \tag{1.3}$$

As the casings are non-deformable

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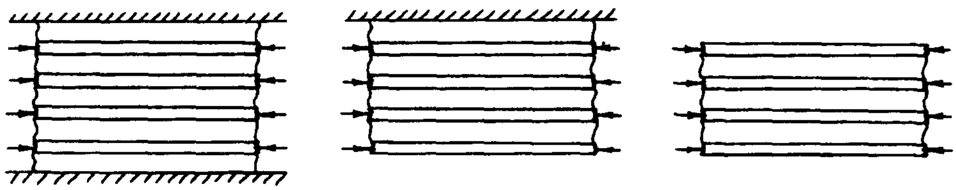

Fig. 1.

$$w_0 = w_{n+1} = 0 \tag{1.4}$$

and for the lowest and highest plates we have

$$q_1^- = -\beta w_1, \quad q_n^+ = -\beta w_n \tag{1.5}$$

Substituting (1.2) into (1.1) using (1.4) we obtain the system of governing equations

$$Dw_i'''' + Tw_i'' + \beta(-w_{i-1} + 2w_i - w_{i+1}) = 0 \tag{1.6}$$

This is a linear homogeneous system with constant coefficients. A non-trivial solution corresponds to loss of stability (corrugation) of the plates. For convenience we divide each equation by the non-zero factor β and write out the characteristic matrix of the system

$$\begin{Vmatrix} \zeta & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & \zeta & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & \zeta & -1 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & 0 & -1 & \zeta & -1 & 0 \\ 0 & \dots & 0 & 0 & -1 & \zeta & -1 \\ 0 & \dots & 0 & 0 & 0 & -1 & \zeta \end{Vmatrix}, \quad \zeta = \frac{D}{\beta}\lambda^4 + \frac{T}{\beta}\lambda^2 + 2 \tag{1.7}$$

We temporarily ignore the connection between ζ and λ and consider the matrix (1.7) as a function of a complex variable ζ and find all the roots ζ_k of the equation $d_n = 0$ in which d_n is the determinant of the $(n \times n)$-matrix (1.7). Expanding along the first (or last) column (or row) we obtain the recurrence relation

$$d_n = \zeta d_{n-1} - d_{n-2} \tag{1.8}$$

where d_{n-1}, d_{n-2} are determinants of similar matrices given by their subscripts. The technique used below works for any recurrence of the form (1.8) and is explained, for example, in [4].

Suppose we have rewritten (1.8) in the form

$$d_n = (a + b)d_{n-1} - abd_{n-2} \tag{1.9}$$

i.e. we have succeeded in finding two complex numbers a and b such that

$$\zeta = a + b, \quad ab = 1 \tag{1.10}$$

Then, regrouping the terms in (1.9) by two methods, we obtain

$$(d_n - ad_{n-1}) = b(d_{n-1} - ad_{n-2}), \quad (d_n - bd_{n-1}) = a(d_{n-1} - bd_{n-2}) \tag{1.11}$$

Considering the (1.11) as recurrence relations (geometric progressions) for the differences contained in the brackets, and applying these relations repeatedly, we arrive at the formulae

$$(d_n - ad_{n-1}) = b^{n-2}(d_2 - ad_1),\quad (d_n - bd_{n-1}) = a^{n-2}(d_2 - bd_1) \tag{1.12}$$

All the quantities on the right-hand sides of (1.12) can be easily expressed in terms of ζ

$$d_2 = \zeta^2 - 1,\quad d_1 = \zeta \tag{1.13}$$

and a and b are roots of the quadratic equation

$$X^2 - \zeta X + 1 = 0 \tag{1.14}$$

When $a \neq b$, eliminating d_{n-1} from (1.12), we obtain

$$d_n = (a^{n-1}(d_2 - bd_1) - b^{n-1}(d_2 - ad_1))/(a - b) \tag{1.15}$$

Substituting (1.10) into (1.13), from (1.15) we obtain the formula

$$d_n = (a^{n+1} - b^{n+1})/(a - b) \tag{1.16}$$

Below it is convenient to use exponential notation for the complex numbers

$$a = |a| e^{i\varphi},\quad b = |b| e^{i\psi} \tag{1.17}$$

where φ and ψ are the principal values of the argument in the complex plane with a cut along the negative semi-axis. The condition $ab = 1$ gives

$$|a| = 1/|b|,\quad \psi = -\varphi \tag{1.18}$$

Computing powers of the right-hand side of (1.17), substituting into (1.16) and using Euler's formula and $\psi = -\varphi$, we obtain

$$d_n = \frac{(|a|^{n+1} - |b|^{n+1})\cos(n+1)\varphi + i(|a|^{n+1} + |b|^{n+1})\sin(n+1)\varphi}{(|a| - |b|)\cos\varphi + i(|a| + |b|)\sin\varphi} \tag{1.19}$$

The denominator vanishes only when $|a| = |b|$ and $\sin\varphi = 0$. The numerator vanishes only when $|a| = |b|$ and $\sin(n + 1)\varphi = 0$. Thus, comparing this with the first formula of (1.19), we conclude that d_n vanishes only when

$$|a| = |b| = 1,\quad \sin(n + 1)\varphi = 0,\ \sin\varphi \neq 0 \tag{1.20}$$

from which it follows that

$$\varphi = \frac{k\pi}{n+1},\quad k = \pm 1, \pm 2, \ldots, \pm n \tag{1.21}$$

irrespective of the values of ζ, and the numbers a and b are complex conjugate. From the condition $\zeta = a + b$ it therefore follows that

$$\zeta = 2\mathrm{Re}a = 2\mathrm{Re}b \tag{1.22}$$

or, using Euler's formula and (1.21)

$$\zeta_k = 2\cos\frac{k\pi}{n+1},\quad k = 1, \ldots, n \tag{1.23}$$

Formula (1.23) gives exactly n different roots of the equation $d_n = 0$. Because d_n is a polynomial in ζ of degree n there are no other roots. The case $a = b$, omitted previously, need not be considered, although such consideration presents no difficulty and leads to an equation without roots.

We now turn to relation (1.7) between ζ and λ. Substituting the values of ζ_k from (1.23) in place of

ζ in these formulae, we obtain n independent biquadratic equations for λ

$$\frac{D}{\beta}\lambda^4 + \frac{T}{\beta}\lambda^2 + 2 - \zeta_k = 0, \quad k = 1, \ldots, n \tag{1.24}$$

For any $k = 1, \ldots, n$ the free term on the left-hand side of (1.24) is positive. Hence, from an analysis of the movement of roots in the complex plane as T increases from zero, or from the multiplicity condition for the λ^2 roots, we obtain n equations

$$T_k = 2[D\beta(2 - \zeta_k)]^{1/2}, \quad k = 1, \ldots, n \tag{1.25}$$

It is obvious that the smallest of the n values of T_k is obtained when $k = 1$ and is given by

$$T_*^n = T_1 = 2\left[2D\beta\left(1 - \cos\frac{\pi}{n+1}\right)\right]^{1/2} \tag{1.26}$$

Here the index n means that the packet has n plates, and the asterisk denotes that it is the critical value of the load.

Expressing D and β in terms of the elastic constants and using $E = 2G(1 + \nu)$, we obtain

$$T_*^n = \left[1 - \cos\frac{\pi}{n+1}\right]^{1/2} T_*^1, \quad T_*^1 = 2\left[\frac{1}{3}\frac{G_p}{(1-\nu_p)}\frac{G}{(1-\nu)}\frac{h^3}{H}\right]^{1/2} \tag{1.27}$$

As n increases to infinity the critical force tends to zero. As n changes from 1 to 20 its value decreases by an order of magnitude.

The form of the loss of stability for an arbitrary pack of plates is sinusoidal with semi-wavelength

$$l_* = \frac{\pi}{\lambda_*} = \pi\left[\frac{D}{\beta}\left(1 - \cos\frac{\pi}{n+1}\right)^{-1}\right]^{1/4} \tag{1.28}$$

It is important to emphasize that the period of the sine-wave is the same for all the plates of the pack. They only differ in amplitude. The amplitude distribution with respect to the plate number is completely given by the eigenvector of the matrix (1.7), defined as always apart from a normalizing factor. The components W_j of the eigenvector corresponding to ζ_k are given by the formula

$$W_j^k = \sin\frac{jk\pi}{n+1}, \quad j, k = 1, \ldots, n \tag{1.29}$$

from which it is clear that the sign of all the eigenvector components for ζ_1 are the same. This sort of stability loss is called *in-phase* [5]: concavities lie below concavities and convexities below convexities. The amplitude of the deflection is smaller the nearer the plate is situated to a casing. This is not the case for the higher modes.

Formula (1.28) shows that the greater the number of plates in the pack, the larger the corrugation generated in the packet. The critical length of the wave tends to infinity when the number of plates increases without limit. As n varies from 1 to 20, its value triples.

Decoding D and β, we rewrite (1.28) as follows:

$$l_*^n = \pi\left(1 - \cos\frac{\pi}{n+1}\right)^{1/4}\left[\frac{1}{6}\frac{G}{G_p}\frac{1-\nu_p}{1-\nu}h^3 H\right]^{1/4} \tag{1.30}$$

(the indices have the same meaning as in (1.26)).

If the packet in Fig. 1(a) is thought of as a pack of strips of unit width, the expressions inside the roots in (1.25) and (1.30) should include the factor $1/(1 - \nu^2)$. Its role is unimportant.

2. PROBLEMS WITH QUASI-TOEPLITZ MATRICES

A square matrix M in which the elements m_{ij} depend only on the difference $i-j$ in the indices, is called a *Toeplitz* matrix [6, 7]. The same element occurs in every position along any fixed diagonal of the matrix. Matrix (1.7) is a tridiagonal symmetric Toeplitz polynomial matrix. *Quasi-Toeplitz* matrices [7] differ from Toeplitz matrices in that the first and last elements along the main diagonal can be arbitrary and different from the main element forming that diagonal. Various problems lead to characteristic matrices of this form, for example, when the thickness and elastic properties of the lowest and highest flexible layers are different from the characteristics of the intermediate flexible layers. Below we consider two limiting cases: a packet not attached to one of the casings, and a packet that is free both below and above (Figs 1b and c, respectively). Such problems can be useful in the mechanics of layered composites.

We first consider the problem corresponding to Fig. 1(c). In the notations of Section 1 we have

$$\bar{q}_1 = \overset{+}{q}_n = 0 \tag{2.1}$$

$$\begin{aligned}
&Dw_1'''' + Tw_1'' + \beta(w_1 - w_2) = 0\\
&Dw_i'''' + Tw_i'' + \beta(-w_{i-1} + 2w_i - w_{i+1}) = 0, \quad i = 2,\dots,n-1\\
&Dw_n'''' + Tw_n'' + \beta(w_n - w_{n-1}) = 0
\end{aligned} \tag{2.2}$$

After dividing all the equations by β the characteristic matrix differs from (1.7) only in that the first and last elements of the main diagonal are replaced by $\xi = \zeta - 1$. As before, we introduce complex variables a and b through the conditions

$$\zeta = a + b, \quad \xi = ab \tag{2.3}$$

The determinants of various orders are denoted by d_n. We have

$$\begin{aligned}
&d_1 = \xi, \quad d_2 = \xi\zeta - 1\\
&d_i = \zeta d_{i-1} - d_{i-2}, \quad i = 3,\dots,n-1; \quad d_n = \xi d_{n-1} - d_{n-2}
\end{aligned} \tag{2.4}$$

When $a \neq b$, in accordance with (1.15)

$$d_i = (b^{i-1}(d_2 - ad_1) - a^{i-1}(d_2 - bd_1))/(b-a), \quad i = 3,\dots,n-1 \tag{2.5}$$

From the recurrence relation (2.4) for d_n and using (2.3) we obtain

$$d_n = (b^{n-1}(\xi - a)^2 - a^{n-1}(\xi - b)^2)/(b-a) = (b^{n-1}(b-1)^2 - a^{n-1}(a-1)^2)/(b-a) \tag{2.6}$$

Assuming $|a| = |b|$, from (2.3) we quickly obtain $b = \bar{a}$, and so $b - 1 = \overline{a-1}$, $-\arg b = \arg a \equiv \varphi$, $b - a = -2\,\mathrm{Im}\,a$. From an elementary geometrical consideration of Fig. 2 we establish the important result

$$\chi \equiv \arg(a-1) = \pi - (\pi - \varphi)/2 = (\pi + \varphi)/2 \tag{2.7}$$

and rewrite (2.6) as

$$d_n \doteq \frac{|a-1|^2}{|a|}\frac{\sin(2\chi + (n-1)\varphi)}{\sin\varphi} \tag{2.8}$$

The equation $d_n = 0$ gives the set of angles

$$2\chi + (n-1)\varphi = k\pi, \quad \varphi \neq j\pi, \quad k, j = 0, \pm 1, \pm 2, \dots \tag{2.9}$$

Expressing χ using (2.7), we obtain

$$\varphi = k\pi/n, \quad (k = 0, \pm 1, \pm 2, \dots) \cap (k \neq 0, \pm n, \pm 2n, \dots) \tag{2.10}$$

Then, since $\zeta = a + b = 2\,\mathrm{Re}\,a$, the roots of $d_n = 0$ are

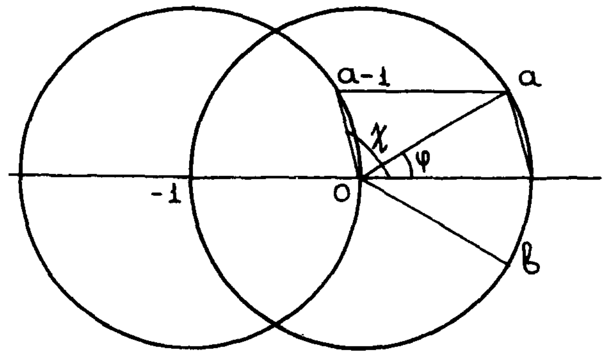

Fig. 2.

$$\zeta_k = 2\cos\frac{k\pi}{n}, \quad k = 1,\dots,n-1 \tag{2.11}$$

and all the other values of k listed in (2.10) give no new values of ζ_k other than those shown in (2.11). The variable $\xi = \zeta - 1$ is a function of the fundamental variable ζ, and d_n is considered to be a polynomial of degree n in ζ over the field of complex numbers. Formula (2.11) contains only $n - 1$ different values of the roots of the polynomial d_n. We find the last root by considering the previously ignored case $a = b$.

When $a = b$ Eqs (1.11) are identical and have the form

$$(d_i - ad_{i-1}) = a(d_{i-1} - ad_{i-2}), \quad i = 3,\dots,n-1 \tag{2.12}$$

Considering (2.12) as a recurrence relation with respect to the expressions in brackets and repeatedly applying it, we obtain

$$d_i - ad_{i-1} = a^{i-2}(d_2 - ad_1) \tag{2.13}$$

We write (2.13) replacing the index i with $i - 1$, and substitute the expression for d_{i-1} into (2.13). We obtain

$$d_i = a^2 d_{i-2} + 2a^{i-2}(d_2 - ad_1) \tag{2.14}$$

Repeating these arguments, we arrive at the formula

$$d_i = (i-1)a^{i-2}d_2 - (i-2)a^{i-1}d_1, \quad i = 3,\dots,n-1 \tag{2.15}$$

Substituting (2.15) with $i = n - 1$, $i = n - 2$ into the recurrence relation (2.4) for d_n, we have

$$d_n = (\xi(n-2) - (n-3)a)a^{n-3}d_2 - (\xi(n-3) - (n-4)a)a^{n-2}d_1 \tag{2.16}$$

When $a = b$ we obtain

$$\xi = 2a - 1, \quad d_1 = 2a - 1, \quad d_2 = 4a^2 - 2a - 1 \tag{2.17}$$

We substitute (2.17) into (2.16) and after reduction we obtain

$$d_n = (a-1)a^{n-3}(n(2a+1)(a-1) + 2(a+1)) \tag{2.18}$$

The equation $d_n = 0$ has a simple root $a = 1$, a root $a = 0$ of multiplicity of $n - 3$ and roots

$$a = (n - 2 \pm [(n-2)(9n-2)]^{1/2}) / 4n \tag{2.19}$$

Of these only $a = 1$ satisfies the condition $a^2 = 1$. From (2.3) it corresponds to the value

$$\zeta_0 = 2 \tag{2.20}$$

All together formulae (2.11) and (2.20) define exactly n different roots of the characteristic equation. We will now analyse the stability of a free packet.

When $n = 1$ the equation

$$D\lambda^4 + T\lambda^2 = 0 \tag{2.21}$$

gives $\lambda^2 = 0$ and $-T = D\lambda^2$. The extremal values are $\lambda_*^2 = 0$, $T_* = 0$. Equilibrium is impossible for any finite force T.

When $n = 2$ the characteristic equation $\xi^2 - 1 = 0$ gives roots $\xi_1 = \xi_2 = -1$. The first of these leads to Eq. (2.21), from which $\lambda_*^2 = T_* = 0$. The second gives the equation

$$D\lambda^4 + T\lambda^2 + 2\beta = 0 \tag{2.22}$$

leading to the critical values

$$\lambda_*^2 = (2\beta / D)^{1/2}, \quad T_* = 2(2D\beta)^{1/2} \tag{2.23}$$

As before equilibrium is impossible for a non-zero force T, but a non-trivial "second mode" exists.

For $n = 3$ or more, writing (2.11) and (2.20) as one formula (2.11) with indices $k = 0, \dots, n-1$, we obtain n equations

$$D\lambda^4 + T\lambda^2 + 2\beta\left(1 - \cos\frac{k\pi}{n}\right) = 0, \quad k = 0,\dots,n-1 \tag{2.24}$$

from which, by a standard procedure, we determine

$$\lambda_*^2 = \left[\frac{2\beta}{D}\left(1 - \cos\frac{k\pi}{n}\right)\right]^{1/2}, \quad T_* = 2\left[2D\beta\left(1 - \cos\frac{k\pi}{n}\right)\right]^{1/2} \tag{2.25}$$
$$k = 0,\dots,n-1$$

The brackets in the radicals vanish when $k = 0$, hence equilibrium is impossible for any $T \neq 0$.

All these conclusions support physical intuition and well-known results for an infinite beam.

Substituting ζ_j instead of ζ into the characteristic matrix of system (2.2), we obtain n degenerate numerical matrices. Each of these annihilates some numerical vector $(W_1^j, \dots, W_n^j)^{\mathrm{T}}$ which we call an eigenvector and whose components we now determine.

For $\zeta_0 = 2$ the solution is obvious

$$W_i^0 = 1, \; i = 1, \dots, n \tag{2.26}$$

When $j = 1, \dots, n-1$ we seek the components W_i^j in the form

$$W_i^j = A\sin i\frac{j\pi}{n} + B\cos i\frac{j\pi}{n}, \quad i = 1,\dots,n \tag{2.27}$$

with undetermined coefficients A and B. Substituting (2.27) into the jth matrix, we see, after reduction, that the second to the $(n-1)$th equations are satisfied identically for all A and B. The first and last equations are identical and give the ratio

$$\frac{B}{A} = \operatorname{ctg}\frac{j\pi}{2n} \tag{2.28}$$

From this, substituting (2.28) into (2.27), we obtain (apart from normalization)

$$W_i^j = \cos\frac{(2i-1)j}{2n}\pi, \quad i = 1,\dots,n, \quad j = 1,\dots,n-1 \tag{2.29}$$

W_i^j is the amplitude of the corrugation of the ith plate in the jth mode for a packet of n plates. The shapes of these stability-loss modes for some n and j are shown in Fig. 3.

Thus the case of a free packet has been completely investigated analytically. All the critical parameters and corresponding corrugation shapes are given by explicit formulae using elementary functions.

We will now consider the case of the packet shown in Fig. 1(b), which is only attached from above to an undeformable casing. We then have

$$q_1^- = 0, \quad w_{n+1} = 0 \tag{2.30}$$

The first equation in (2.2) still holds, as does the second when $i = 2, \dots, n$. The characteristic matrix differs from (1.7) only in that the first element of the main diagonal is $\xi = \zeta - 1$. When $a \neq b$, we find, as for (2.8), that

$$d_n = \frac{|a-1|}{|a|}\frac{\sin(\chi + n\varphi)}{\sin\varphi} \tag{2.31}$$

From this, like (2.10)

$$\varphi = \frac{2k-1}{2n+1}\pi, \quad (k = 0, \pm1, \pm2, \dots) \cap \left(\frac{2k-1}{2n+1} \neq 0, \pm1, \pm2, \dots\right) \tag{2.32}$$

Like (2.11) we have

$$\zeta_k = 2\cos\frac{2k-1}{2n+1}\pi, \quad k = 1, \dots, n \tag{2.33}$$

Unlike the previous case, the list of values of k in (2.32) is restricted on account of φ falling into the interval $(-\pi, \pi)$. Formula (2.33) contains n different roots of the equation $d_n = 0$. The case $a = b$ leads to the polynomial

$$d_n = a^{n-2}((2a^2 - a - 1)n + 1) \tag{2.34}$$

which has no roots a such that $a^2 = 1$.

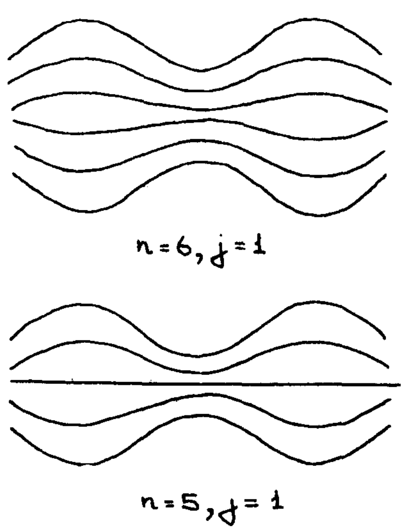

Fig. 3.

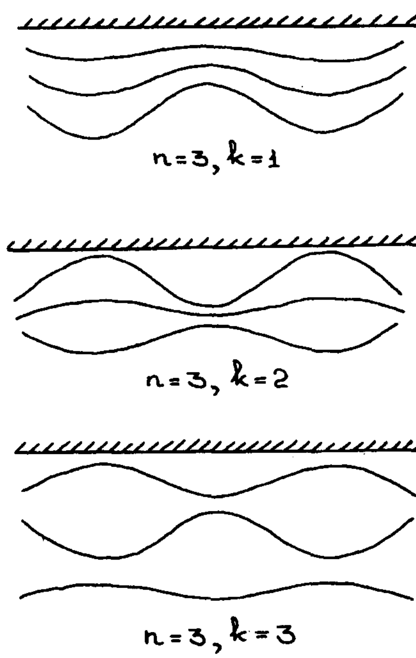

Fig. 4.

Substituting (2.33) into the expressions for ξ and ζ in terms of λ, the usual reduction gives

$$\lambda_* = \left[\frac{2\beta}{D}\left(1-\cos\frac{2k-1}{2n+1}\pi\right)\right]^{1/4}, \quad T_* = 2\left[2D\beta\left(1-\cos\frac{2k-1}{2n+1}\pi\right)\right]^{1/2}$$
$$k = 1,\ldots,n \tag{2.35}$$

The minimum values of T_* with respect to k is reached at $k = 1$. Comparing this with (1.26) we verify that it is *smaller* than for a packet fixed at both sides. The corresponding wave number is also smaller, i.e. the wavelength is *larger* than in the original Toeplitz case.

The eigenvector corresponding to ζ_k is sought in the form (2.27) and has components

$$W_i^k = \cos\frac{(2i-1)(2k-1)}{2(2n+1)}\pi, \quad i,k = 1,\ldots,n \tag{2.36}$$

The first mode $k = 1$ is "in-phase". Its corrugation amplitude is larger the nearer the plate is to the free surface. For higher modes this property is lost. For particular numbers of layers n the first and some other modes are shown in Fig. 4. It is interesting to note that the set of amplitude moduli for all nodes at fixed n is the same. Moreover, it is identical with the set of amplitudes of the problem for the free packet with odd n (the formulae for the problem in Fig. 1(b) can be obtained from similar formulae for the problem in Fig. 1(c) when n and j are replaced by $2n + 1$ and $2k - 1$ respectively).

Thus this case of "surface-layer stability" has been also completely investigated by simple analytic methods and leads to simple formulae.

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